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Second and third order observables of the two-matrix model*

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Abstract: In this paper we complement our recent result on the explicit formula for the planar limit of the free energy of the two-matrix model by computing the second and third order observables of the model in terms of canonical structures of the underlying genus g spectral curve. In particular we provide explicit formulas for any three-loop correlator of the model. Some explicit examples are worked out.

Keywords: Integrable Hierarchies, Matrix Models.

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Contents

1. Introduction

The present paper is the natural complement of our recent publication [1] in which an explicit residue formula for the free-energy in the planar limit (and for arbitrary genus spectral curves) of the hermitean two-matrix model was derived. In point of fact the present results, while logically consequent that paper, can be (and in fact are) derived without using the explicit formula in [1]. Given the well-known relation of the free-energy of this model with the tau-function of the dispersionless Toda hierarchy, we could as well rephrase the entire contents of this (and the previous) paper in the language of integrable hierarchies.

Let us briefly recall the historical and logical setting of the problem. The object under scrutiny is the random 2-matrix model [2, 3], which has many applications to solid state physics [4] (e.g., conduction in mesoscopic devices, quantum chaos and, lately, crystal growth[5]), particle physics [6], $2d$-quantum gravity and string theory [7]–[9]. The model consists of two hermitean matrices M_1, M_2 of size $N \times N$ with a probability distribution given by the formula

$$\begin{aligned} \mathrm{d}\mu(M_1, M_2) &= \frac{1}{\mathcal{Z}_N} \mathrm{d}M_1 \mathrm{d}M_2 \exp\left[-\frac{1}{\hbar}\mathrm{Tr}\left(V_1(M_1) + V_2(M_2) - M_1 M_2\right)\right], \\ V_1(x) &= \sum_{K=1}^{\infty} \frac{u_K}{K} x^K ; \qquad V_2(y) = \sum_{J=1}^{\infty} \frac{v_J}{J} y^J , \end{aligned} \tag{1.1}$$

where the functions V_i (called *potentials*) are polynomials of degree $d_i + 1$ for simplicity (and definiteness), but could be formally extended to power series. The partition function $\mathcal{Z}_N$ is known to be a τ-function for the KP hierarchy in each set of deformation parameters (coefficients of V_1 or V_2) and to provide solutions of the two-Toda hierarchy [10]–[12]. This model with polynomial potentials has been investigated in the series of paper [13]–[15] where a duality of spectral curves and differential systems for the relevant biorthogonal polynomials has been unveiled and analyzed in the case of polynomial potentials. In [16] the mixed correlation functions of the model (traces of powers of the two non-commuting matrices) have been reduced to a formal Fredholm-like determinant without any assumption on the nature of the potentials and using the recursion coefficients for the biorthogonal polynomials. We recall that the biorthogonal polynomials are two sequences of monic polynomials ([13] and references therein)

$$\pi_n(x) = x^n + \cdots, \qquad \sigma_n(y) = y^n + \cdots, \qquad n = 0, 1, \dots \tag{1.2}$$

that are "orthogonal" (better say "dual") w.r.t. to the coupled measure on the product space

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}y\ \pi_n(x)\sigma_m(y)\mathrm{e}^{-\frac{1}{\hbar}(V_1(x)+V_2(y)-xy)} = h_n\delta_{mn}, \qquad h_n \neq 0\ \forall n \in \mathbb{N}, \tag{1.3}$$

where $V_1(x)$ and $V_2(y)$ are the functions in the measure (1.1). It is convenient to introduce the associated quasipolynomials defined by the formulas

$$\begin{aligned} \psi_n(x) &:= \frac{1}{\sqrt{h_{n-1}}}\pi_{n-1}(x)\mathrm{e}^{-\frac{1}{\hbar}V_1(x)} \\ \phi_n(y) &:= \frac{1}{\sqrt{h_{n-1}}}\sigma_{n-1}(y)\mathrm{e}^{-\frac{1}{\hbar}V_2(y)}\,. \end{aligned} \tag{1.4}$$

In terms of these two sequences of quasipolynomials the multiplications by x and y respectively are represented by semiinfinite square matrices $Q = [Q_{ij}]_{i,j\in\mathbb{N}^*}$ and $P = [P_{ij}]_{i,j\in\mathbb{N}^*}$ according to the formulae

$$\begin{aligned} x\psi_n(x) &= \sum_m Q_{n,m}\psi_m(x)\,; \qquad y\phi_n(y) = \sum_m P_{m,n}\phi_m(y) \\ Q_{n,m} &= 0 = P_{m,n}\,, \qquad \text{if } n > m+1\,. \end{aligned} \tag{1.5}$$

The matrices P and Q satisfy the "string equation"

$$[P, Q] = \hbar\mathbb{1}\,. \tag{1.6}$$

We refer for further details to [17] and [13]–[15] where these models are studied especially in the case of polynomial potentials. We also point out that the model can easily be generalized to accommodate contours of integration other than the real axes and arbitrary (possibly complex) potentials [17, 13].

The partition function is believed to have a large N expansion (with $\hbar = t/N$ and t fixed) according to the formula

$$-\frac{1}{N^2}\ln\mathcal{Z}_N = \mathcal{F} = \mathcal{F}^{(0)} + \frac{1}{N^2}\mathcal{F}^{(1)} + \cdots\,. \tag{1.7}$$

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This expansion in powers of N^{-2} has been repeatedly advocated for the 2-matrix model on the basis of physical arguments [8, 18, 19] and has been rigorously proven in the one-matrix model [20]. In the two-matrix model this expansion is believed to generate 2-dimensional statistical models of surfaces triangulated with ribbon-graphs [8, 7, 21], where the powers of N^{-1} are the Euler characteristics of the surfaces being tessellated. From this point of view the term $\mathcal{F}^{(0)}$ corresponds to a genus 0 tessellation and the next to a genus one tessellation. Remarkably, an algorithm for the computation of the subleading terms in the $\hbar = \frac{t}{N}$ expansion is known and also a closed expression of the genus 1 correction $\mathcal{F}^{(1)}$ [18, 19].

The object of [1] was the leading term of the free energy, $\mathcal{F}^{(0)}$. It is the generating function (to leading order) of the expectations of the powers of the two matrices in the model

$$\left\langle \mathrm{Tr}\, {M_1}^K \right\rangle = K\partial_{u_K}\mathcal{F}^{(0)} + \mathcal{O}\left(N^{-2}\right), \qquad \left\langle \mathrm{Tr}\, {M_2}^J \right\rangle = J\partial_{v_J}\mathcal{F}^{(0)} + \mathcal{O}\left(N^{-2}\right). \tag{1.8}$$

It is at the same time the logarithm of the τ-function of the dispersionless Toda hierarchy (in the one-cut case) [22, 23]; this hierarchy falls in a broad sense in the same sort of integrable dispersionless hierarchies called "universal Whitham hierarchy" (i.e. dispersionless KdV and KP) and studied in [24], where formulas for the corresponding τ-function are provided which are quite similar to that in [1]. However the situation under inspection in this paper (and in [1]) involves a hierarchy which does not quite fall in the very broad class of [24]; there the moduli space under consideration was the set of curves of genus g with N marked points ∞_k, $k = 1, \ldots, N$ and local parameters around the punctures, together with a fixed meromorphic differential of the second kind $\mathrm{d}\mathcal{E}$ with poles of degree $n_k + 1$ at ∞_k. The (local) dimension of the space of these data is $3g - 2 + \sum_{k=1}^{N}(n_k + 1)$.

The object of interest of the present paper is -among others- the explicit computation of the double and triple correlators

$$\left\langle \mathrm{Tr}\, {M_i}^a\, \mathrm{Tr}\, {M_j}^b \right\rangle_{\mathrm{conn}}, \qquad \left\langle \mathrm{Tr}\, {M_i}^a\, \mathrm{Tr}\, {M_j}^b\, \mathrm{Tr}\, {M_k}^c \right\rangle_{\mathrm{conn}}, \quad i,j,k = 1,2,\ a,b,c \in \mathbb{N}. \tag{1.9}$$

or — which is the same — their generating functions

$$\begin{aligned}
\left\langle \mathrm{Tr}\,\frac{1}{x_1 - M_1}\,\mathrm{Tr}\,\frac{1}{x_2 - M_1}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_1(x_1)}\frac{\delta}{\delta V_1(x_2)}\mathcal{F} \\
\left\langle \mathrm{Tr}\,\frac{1}{x - M_1}\,\mathrm{Tr}\,\frac{1}{y - M_2}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_1(x)}\frac{\delta}{\delta V_2(y)}\mathcal{F} \\
\left\langle \mathrm{Tr}\,\frac{1}{y_1 - M_2}\,\mathrm{Tr}\,\frac{1}{y_2 - M_2}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_2(y_1)}\frac{\delta}{\delta V_2(y_2)}\mathcal{F},
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
\left\langle \mathrm{Tr}\,\frac{1}{x_1 - M_1}\,\mathrm{Tr}\,\frac{1}{x_2 - M_1}\,\mathrm{Tr}\,\frac{1}{x_3 - M_1}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_1(x_1)}\frac{\delta}{\delta V_1(x_2)}\frac{\delta}{\delta V_1(x_3)}\mathcal{F} \\
\left\langle \mathrm{Tr}\,\frac{1}{x_1 - M_1}\,\mathrm{Tr}\,\frac{1}{x_2 - M_1}\,\mathrm{Tr}\,\frac{1}{y - M_2}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_1(x_1)}\frac{\delta}{\delta V_1(x_2)}\frac{\delta}{\delta V_2(y)}\mathcal{F} \\
\left\langle \mathrm{Tr}\,\frac{1}{y_1 - M_2}\,\mathrm{Tr}\,\frac{1}{y_2 - M_2}\,\mathrm{Tr}\,\frac{1}{x - M_1}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_2(y_1)}\frac{\delta}{\delta V_2(y_2)}\frac{\delta}{\delta V_1(x)}\mathcal{F} \\
\left\langle \mathrm{Tr}\,\frac{1}{y_1 - M_2}\,\mathrm{Tr}\,\frac{1}{y_2 - M_2}\,\mathrm{Tr}\,\frac{1}{y_3 - M_2}\right\rangle_{\mathrm{conn}} &= \frac{\delta}{\delta V_2(y_1)}\frac{\delta}{\delta V_2(y_2)}\frac{\delta}{\delta V_2(y_3)}\mathcal{F},
\end{aligned} \tag{1.11}$$

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where the "puncture" operators are defined (a bit formally) as

$$\frac{\delta}{\delta V_1(x)} := \sum_{K=1}^{\infty} x^{-K-1} K \partial_{u_K}\,, \qquad \frac{\delta}{\delta V_2(y)} := \sum_{J=1}^{\infty} y^{-J-1} J \partial_{v_J}\,. \tag{1.12}$$

The meaning of the puncture operators when the potentials are polynomials is given by the understanding that each derivative — say — w.r.t. u_K gives a $\mathrm{Tr}\, {M_1}^K$ contribution inside the expectation value. The two-loops correlators (1.10) are well studied and our results here are not new at least in the one-cut case [25], but are needed for computing the three-loop correlator (1.11). The latter have not been computed for this model; the similar problem for the normal matrix model has been addressed in [26] in a setting that is a subsetting of our, primarily due to the fact their case corresponds to a spectral curve of genus 0 plus certain reality conditions. Indeed the planar limit of the free energy $\mathcal{F}^{(0)}$ (we will drop the superscripts from now on) can be defined for any spectral curve Σ_g of genus g as a closed differential on a certain moduli space and this differential can be integrated explicitly ([1] and references therein). In the literature on matrix models the genus of this spectral curve is related to the number of connected components (called "cuts") of the support of the densities of the spectra of the two matrices; in this terminology the case of genus 0 spectral curves corresponds to the *one cut* assumption, whereas the higher genus case to multicut solutions.

In the present paper we complement our previous paper [1] by computing explicitly the second and third order derivatives of the free energy. As we have anticipated, however, these result are derived independently of [1], only the notation being borrowed thence. We also point out that the resulting formulas are very close to similar residue formulas in [24] which, however, deals with a different hierarchy and moduli space and whose proofs are not given explicitly therein.

The paper is organized as follows: in section 2 we set up the problem and notation, recalling the formula [1] for the free energy $\mathcal{F}_g$ over a spectral curve of genus g. We also link our present results to the current relevant literature. In section 2.1 we give a to-the-bone review of the Bergman kernel on algebraic curves and of the properties that will be used in the sequel. In section 2.2 we compute all second order observables, which include the derivatives with respect to the filling fraction and the "temperature" t (which can be interpreted as number operator as well depending on the points of view). Finally, in section 3 we compute all third order observables by deriving a formula for the variations of the conformal structure of the spectral curve under variation of the parameters of the problem.

2. Setting and notations

As in our previous paper [1] we will work with the following data: a (smooth) curve Σ_g of genus g with two marked points ∞_Q, ∞_P and two functions P and Q which have the following pole structure;

1. The function Q has a simple pole at ∞_Q and a pole of degree d_2 at ∞_P.

2. The function P has a simple pole at ∞_P and a pole of degree d_1 at ∞_Q.

From these data it would follow that P, Q satisfy a polynomial relation and hence that the curve is a plane algebraic (singular) curve, but we will not need it for our computations.[1] Moreover we will fix a symplectic basis of cycles $\{a_j, b_j\}_{j=1,\dots g}$ in the homology of the curve. By their definition we have

$$
\begin{aligned}
P &= \sum_{K=1}^{d_1+1} u_K Q^{K-1} - \frac{t}{Q} - \sum_{K=-1}^{-\infty} K U_K Q^{-K-1} =: V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}) , \quad \text{near } \infty_Q \\
Q &= \sum_{J=1}^{d_2+1} v_J P^{K-1} - \frac{t}{P} - \sum_{J=-1}^{-\infty} J V_J P^{-J-1} =: V_2'(P) - \frac{t}{P} + \mathcal{O}(P^{-2}) , \quad \text{near } \infty_P . \qquad (2.1)
\end{aligned}
$$

The fact that the coefficient of the power Q^{-1} or P^{-1} is the same follows immediately from computing the sum of the residues of $P\mathrm{d}Q$ (or $Q\mathrm{d}P$). The polynomials V_1 and V_2 defined by the above formula (2.1) are the potentials of the matrix model whose free energy we are considering[2] and the functions Q and P represent the semiclassical (commutative as per eq. (1.6)) limit of the multiplication operators for the orthogonal polynomials. The coefficients u_K, v_J, t are read off eqs. (2.1) as follows

$$
u_K = -\operatorname*{res}_{\infty_Q} P Q^{-K} \mathrm{d}Q , \qquad v_J = -\operatorname*{res}_{\infty_P} Q P^{-J} \mathrm{d}P , \qquad t = \operatorname*{res}_{\infty_Q} P\mathrm{d}Q = \operatorname*{res}_{\infty_P} Q\mathrm{d}P . \qquad (2.3)
$$

Note that the requirement that the curve possesses two meromorphic functions with this pole structure imposes strong constraints on the moduli of the curve itself. In fact a Riemann-Roch argument shows that the moduli space of these data is of dimension $d_1 + d_2 + 3 + g$; to the above $(d_1 + 1) + (d_2 + 1) + 1$ parameters we add the following period integrals referred to as *filling fractions*

$$
\epsilon_i := \frac{1}{2i\pi} \oint_{a_i} P\mathrm{d}Q , \qquad i = 1, \dots g . \qquad (2.4)
$$

Here we have introduced a symplectic basis $\{a_i, b_i\}_{i=1\dots g}$ in the homology of the curve Σ_g and the choice of the a-cycles over the b-cycles is purely conventional.

The free energy $\mathcal{F}_g$ of the model (where the subscript refers to the genus of the spectral curve, not the genus of the $\hbar$ expansion) is then defined by the equations

$$
\begin{aligned}
\partial_{u_K} \mathcal{F}_g &= \frac{1}{K} \operatorname*{res}_{\infty_Q} P Q^K \mathrm{d}Q = U_K , \qquad \partial_{v_J} \mathcal{F}_g = \frac{1}{J} \operatorname*{res}_{\infty_P} Q P^J \mathrm{d}P = V_J , \\
\partial_{\epsilon_i} \mathcal{F}_g &= \oint_{b_j} P\mathrm{d}Q =: \Gamma_i . \qquad (2.5)
\end{aligned}
$$

[1] We could also generalize this setting to the case $d_1 = d_2 = \infty$ which would simply mean that the two functions have one simple pole and one essential singularity; this generalization is quite straightforward and would need only some care about convergence.

[2] Formula (2.1) defines only their derivative; what we mean by $V_1(x)$ and $V_2(y)$ is explicitly

$$
V_1(x) := \sum_{K=1}^{d_1+1} \frac{u_K}{K} x^K , \qquad V_2(y) := \sum_{J=1}^{d_2+1} \frac{v_J}{J} y^J . \qquad (2.2)
$$

The constant term in the two potential would have trivial consequences as it amounts to a rescaling of the partition function of the matrix model by $\mathrm{e}^{u_0+v_0}$, which of course has absolutely no consequences on all the observables of the model.

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These equations and in particular the derivatives with respect to the filling fractions appear in this precise context in [27]; nevertheless, after identifying the free energy with some tau function for a dispersionless Whitham hierarchy, similar formulas appear in [24].

The above equations mean that the coefficients of singular parts of P *qua* function of Q (or vice-versa) are the independent coordinates and the coefficients of the regular part (starting from Q^{-2}) are the corresponding derivatives of the free energy. Moreover the a-periods of $P\mathrm{d}Q$ are the independent variables and the b-periods the corresponding derivatives of $\mathcal{F}_g$. In fact one should add the extra constraints $\Gamma_i = 0$ to this functional in order to assure that it comes from a saddle-point integration of the two-matrix model. However, since we are interested here mostly with the formal $N \to \infty, \hbar \to 0,\ \hbar N = t$ limit (as explained in [28]) we do not impose the extra constraint and treat the filling fractions as independent coordinates.

It was shown in [1] (but see also [28] for the extension to the chain of matrices) that $\mathcal{F}_g$ is given by the following equivalent formulas

$$\begin{aligned} 2\mathcal{F}_g &= \operatorname*{res}_{\infty_Q} P\Phi_1 \mathrm{d}Q + \operatorname*{res}_{\infty_P} Q\Phi_2 \mathrm{d}P + \frac{1}{2}\operatorname*{res}_{\infty_Q} P^2 Q \mathrm{d}Q + t\mu + \sum_{i=1}^{g} \epsilon_i \Gamma_i \\ &= \sum_K u_K U_K + \sum_J v_J V_J + \frac{1}{2}\operatorname*{res}_{\infty_Q} P^2 Q\mathrm{d}Q + t\mu + \sum_{i=1}^{g}\epsilon_i\Gamma_i \\ &= \sum_K \frac{2-K}{2} u_K U_K + \sum_J \frac{2-J}{2} v_J V_J + t\mu + \sum_{i=1}^{g}\epsilon_i\Gamma_i - \frac{1}{2}t^2 \,, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Phi_1(\zeta) &:= \frac{1}{2i\pi}\oint_{\infty_Q} \ln\left(1 - \frac{Q}{Q(\zeta)}\right) P\mathrm{d}Q \\ &= -V_1(Q(\zeta)) + t\ln(Q(\zeta)) + \int_{X_q}^{\zeta} P\mathrm{d}Q = \sum_{K=1}^{\infty} U_K Q^{-K} = \mathcal{O}(Q^{-1}) \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Phi_2(\zeta) &:= \frac{1}{2i\pi}\oint_{\infty_P} \ln\left(1 - \frac{P}{P(\zeta)}\right) Q\mathrm{d}P \\ &= -V_2(P(\zeta)) + t\ln(P(\zeta)) + \int_{X_p}^{\zeta} Q\mathrm{d}P = \sum_{J=1}^{\infty} V_J P^{-J} = \mathcal{O}(P^{-1}) \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mu &:= \operatorname*{res}_{\infty_Q}\left[V_1(Q) - t\ln\left(\frac{Q}{\lambda}\right)\right]\mathrm{d}S - \operatorname*{res}_{\infty_P}\left[V_2(P) - t\ln(P\lambda)\right]\mathrm{d}S - \\ &\quad - \operatorname*{res}_{\infty_Q} PQ\mathrm{d}S + \sum_{i=1}^{g}\epsilon_i \oint_{b_i}\mathrm{d}S \,, \end{aligned} \tag{2.9}$$

where $\mathrm{d}S$ is the normalized differential of the third kind with poles at $\infty_{P,Q}$ and residues $\operatorname*{res}_{\infty_Q}\mathrm{d}S = -1 = -\operatorname*{res}_{\infty_P}\mathrm{d}S$ and the function λ is the following function (defined up to a multiplicative constant which drops out of the above formula) on the universal covering of the curve with a simple zero at ∞_Q and a simple pole at ∞_P

$$\lambda := \exp\left(\int \mathrm{d}S\right). \tag{2.10}$$

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Remark 1. *The "chemical potential" μ in the context of the dispersionless Toda hierarchy is the long-wave limit of the Toda field [29] (denoted by ϕ there) and satisfies, among others, the Toda field equation which is written, in our notations,*

$$\frac{\partial^2 \mu}{\partial u_1 \partial v_1} + \partial_t \exp(\partial_t \mu) = 0\,. \tag{2.11}$$

Remark 2. *In formulas (2.6),…,(2.9) the open paths of integration are supposed to not intersect the basis of cycles, i.e., to remain in the fundamental cell of the universal covering of the curve and the residues involving the multivalued function λ are taken to be on the same cell of the universal cover.*

We now make the necessary connections with the relevant literature [30]–[35] and [22, 27].

The equations that define here the free energy in the genus $g = 0$ case are precisely the same that define the τ-function of the dToda hierarchy where one imposes the (compatible) constraint of the string equation to the Lax functions. In the relevant literature [22, 23, 36, 37] the functions P, Q are the Lax operators denoted by $L, \tilde{L}^{-1}$ or $\mathcal{L}, \tilde{\mathcal{L}}^{-1}$ and the normalization is slightly different. In higher genus our free energy is related to the tau function for solutions obtained via a Whitham averaging method on some invariant submanifold of the hierarchy.

The link with the important works [30] and [32]–[35] is as follows; if the two potentials are complex conjugate $V_1 = \overline{V_2} =: V$ then (in genus 0) the functions Q and P are conjugate-Schwarz-inverted in the sense explained presently; pick the uniformizing parameter λ of the rational curve Σ_0 to have a zero at ∞_P and a pole at ∞_Q(and a suitable normalization)[3]. Then one has

$$Q(\lambda) = \overline{P(\overline{\lambda}^{-1})}\,. \tag{2.12}$$

The function $Q(\lambda)$ is then the uniformizing map of a Jordan curve Γ in the Q-plane (at least for suitable ranges of the parameters) which is defined by either of the following relations

$$\overline{Q(\lambda)} = P\left(\frac{1}{\lambda}\right) \quad \text{or} \quad |\lambda| = 1\,. \tag{2.13}$$

In the setting of [33, 30, 32] the function Q is denoted by z (and λ by w) so that then P is the Schwartz function of the curve Γ, defined by

$$\overline{z} = S(z)\,, \qquad z \in \Gamma\,. \tag{2.14}$$

The coefficients of the potential $V(x) = \sum_K \frac{t_K}{K} x^K$ are the so-called "exterior harmonic moments" of the region $\mathcal{D}$ enclosed by the curve Γ

$$\begin{aligned} t_K &= \frac{1}{2i\pi} \oint_\Gamma \overline{z} z^{-K} \mathrm{d}z \\ t = t_0 &= \frac{1}{2i\pi} \iint_{\mathcal{D}} \mathrm{d}z \wedge \mathrm{d}\overline{z} = \frac{1}{2i\pi} \oint_\Gamma \overline{z} \mathrm{d}z = \frac{\mathrm{Area}(\mathcal{D})}{\pi}\,. \end{aligned} \tag{2.15}$$

[3] This λ is exactly the same λ appearing in the higher genus formulas; in fact — quite obviously — $\lambda = \exp \int \mathrm{d}\lambda/\lambda$, which is translated in the higher genus setting simply by replacing $\mathrm{d}\lambda/\lambda$ with the third-kind differential $\mathrm{d}S$.

By writing $\overline{z} = S(z(w))$ these integrals become residues in the w-plane. For conformal maps (i.e. Jordan curves) the τ-function has been defined in [35] and given an appealing interpretation as (exponential of the Legendre transform of) the electrostatic potential of a uniform 2-dimensional distribution of charge in $\mathcal{D}$ [30]

$$\ln(\tau_\Gamma) = -\frac{1}{\pi}\iint_{\mathcal{D}} \ln\left|\frac{1}{z} - \frac{1}{z'}\right| \mathrm{d}^2z\mathrm{d}^2z' \ . \tag{2.16}$$

It can be rewritten as a (formal) series in the exterior and interior moments as (note that we are changing the normalization used in [30] and [32]–[34] to match more closely ours)

$$2\ln(\tau_\Gamma) = -\frac{1}{4\pi}\iint_{\mathcal{D}} \mathrm{d}^2z|z|^2 + t_0w_0 + \sum_{K>0}(t_Kw_K + \bar{t}_K\overline{w}_K)\,, \tag{2.17}$$

where the interior moments are defined by (the normalization here differs slightly from [33])

$$\begin{aligned}\partial_{t_k}\tau_\Gamma &= w_K = \frac{1}{\pi K}\iint_{\mathcal{D}} z^K\mathrm{d}^2z\,, \qquad K>0\\ \partial_{t_0}\tau_\Gamma &= w_0 = \frac{1}{\pi}\iint_{\mathcal{D}} \ln|z|^2\mathrm{d}^2z\,.\end{aligned} \tag{2.18}$$

The logarithmic moment w_0 corresponds to our "chemical potential" μ; it is not at all obvious (at least we failed to prove it by direct integration) but it is necessarily so because both $\mathcal{F}_0$ and τ_Γ satisfy the same differential equations w.r.t. t_K (or, in our notation $u_K = \overline{v}_K$) and the same scaling constraint

$$4\mathcal{F} = -t^2 + \left(2t\partial_t + \sum_K(2-K)u_K\partial_{u_K} + \sum_J(2-J)v_J\partial_{v_J}\right)\mathcal{F}\,. \tag{2.19}$$

Remark 3. *Note that formula (2.6) is quite effective in that it allows explicit computations. For instance in the genus zero case one could do the exercise (e.g. with Maple) of explicitly presenting the free energy for arbitrary degrees of the potentials. In the approach of [30] and [32]–[34] the explicit computation in case of regions with finite number of nonzero external moments would be prevented mainly by the analytic computation of the logarithmic moment which, in our approach is nothing but* $\mu = (V_1'(Q)+V_2'(P)-PQ)_0 - t\ln\gamma^2$, *where the subscript* $_0$ *means the constant part in* λ *and* γ *is the "conformal radius" of the domain. This observation leads to the following*

Corollary 1. *Let* $\mathcal{D}$ *be a simply connected domain with finite number (say* $d+1$*) of nonzer external harmonic moments and let*

$$z(w) = \gamma w + \sum_{j=0}^{d}\alpha_j w^{-j}\,, \qquad S(w) = \frac{\gamma}{w} + \sum_{j=0}^{d}\overline{\alpha}_j w^j \tag{2.20}$$

be the Riemann uniformizing map and Schwarz-function respectively. Then

$$\frac{1}{\pi}\iint_{\mathcal{D}} \ln|z|^2\mathrm{d}^2z = (V'(z) + \overline{V'}(S) - zS)_0 - t\ln\gamma^2\,, \tag{2.21}$$

where the subscript $_0$ *means the constant part of the Laurent polynomial in the bracket,* $t = \frac{1}{\pi}Area(\mathcal{D})$ *and* $V(z) = \sum_{K=1}^{d+1}\frac{t_K}{K}z^K$, *with* t_K *the* K*-th external harmonic moment.*

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Recently [31] the above tau-function τ_Γ has been generalized to multiply connected domains. Also this case is contained in our case by imposing the same constraint on the potentials $V_1 = \overline{V_2} = V$. This amounts to saying that the curve admits an antiholomorphic involution φ given (coarsely speaking) by $Q \to \overline{P}$ and interchanging the poles ∞_P and ∞_Q. Such a curve would then become the Schottky double used in [31] and the boundary of the multiply connected domain would be defined implicitly by $Q(\zeta) = \overline{P(\zeta)}$ (which defines — in certain cases — $g+1$ cycles on the curve Σ_g)[4]. The reader familiar with the results in [31] will have no difficulty in finding the necessary dictionary to translate the two settings, provided he/she imposes the restriction on the potentials in our more general case.

2.1 Bergman kernel

The Bergman kernel[5] is a classical object in complex geometry and can be represented in terms of prime forms and Theta functions. In fact we will not need any such sophistication because we are going to use only its fundamental properties (that uniquely determine it). The Bergman kernel $\Omega(\zeta,\zeta')$ (where ζ,ζ' denote here and in the following abstract points on the curve) is a bi-differential on $\Sigma_g \times \Sigma_g$ with the properties

$$\begin{aligned}
&\text{Symmetry:} \qquad \Omega(\zeta,\zeta') = \Omega(\zeta',\zeta)\\
&\text{Normalization:} \qquad \oint_{\zeta'\in a_j} \Omega(\zeta,\zeta') = 0\\
&\oint_{\zeta'\in b_j} \Omega(\zeta,\zeta') = 2i\pi\omega_j(\zeta) = \text{the holomorphic normalized abelian differential}\,.
\end{aligned} \tag{2.22}$$

It is holomorphic everywhere on $\Sigma_g \times \Sigma_g \setminus \Delta$, and it has a double pole on the diagonal $\Delta := \{\zeta = \zeta'\}$: namely, if $z(\zeta)$ is any coordinate, we have

$$\Omega(\zeta,\zeta') \underset{\zeta\sim\zeta'}{\simeq} \left[\frac{1}{(z(\zeta)-z(\zeta'))^2} + \frac{1}{6}S_0(\zeta) + \mathcal{O}(z(\zeta)-z(\zeta'))\right] \mathrm{d}z(\zeta)\mathrm{d}z(\zeta')\,, \tag{2.23}$$

where the very important quantity $S_0(\zeta)$ is the "projective connection" (it transforms like the schwarzian derivative under changes of coordinates).

It follows also from the general theory that any normalized abelian differential of the third kind with simple poles at two points z_- and z_+ with residues respectively ± 1 is obtained from the Bergman kernel as

$$\mathrm{d}S_{z_+,z_-}(\zeta) = \int_{\zeta'=z_-}^{z_+} \Omega(\zeta,\zeta')\,. \tag{2.24}$$

[4]An algebraic curve of genus g with an antiholomorphic involution and $g+1$ real components is called an Harnack's M-curve; in general a curve with the aforementioned properties can have $k \le g+1$ real components, see for instance[38]

[5]Our use of the term "Bergman kernel" is slightly unconventional, since more commonly the Bergman kernel is a reproducing kernel in the L^2 space of holomorphic one-forms. The kernel that we here name "Bergman" is sometimes referred to as the "fundamental symmetric bidifferential". We borrow the (ab)use of the name "Bergman" from [39].

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2.1.1 Prime form

For the sake of completeness and comparison with the results of [31] we recall here that the definition of the prime form $E(\zeta,\zeta')$.

Definition 1. *The prime form $E(\zeta,\zeta')$ is the $(-1/2,-1/2)$ bi-differential on $\Sigma_g\times\Sigma_g$*

$$
\begin{aligned}
E(\zeta,\zeta') &= \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\mathfrak{u}(\zeta)-\mathfrak{u}(\zeta'))}{h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta)h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta')} \\
h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta)^2 &:= \sum_{k=1}^{g}\partial_{\mathfrak{u}_k}\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\Bigg|_{\mathfrak{u}=0}\omega_k(\zeta)\,,
\end{aligned}
\tag{2.25}
$$

where ω_k are the normalized abelian holomorphic differentials, $\mathfrak{u}$ is the corresponding Abel map and $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is a half-integer odd characteristic (the prime form does not depend on which one).

Then the relation with the Bergman kernel is the following

$$
\Omega(\zeta,\zeta') = \mathrm{d}_\zeta\mathrm{d}_{\zeta'}\ln E(\zeta,\zeta') = \sum_{k,j=1}^{g}\partial_{\mathfrak{u}_k}\partial_{\mathfrak{u}_j}\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\Bigg|_{\mathfrak{u}(\zeta)-\mathfrak{u}(\zeta')}\omega_k(\zeta)\omega_j(\zeta')\,. \tag{2.26}
$$

2.2 Second order observables

The second order observables of the two-matrix model (in fact, the multi-matrix model) have already been investigated in the literature [28, 31, 25] and their relation with the Bergman kernel extensively documented. Here we just bring a different and possibly more rigorous derivation of those identities. The main reason why the second order observables appear already in the literature is the expectation (and -in some cases, mostly in the one-matrix model setting- proof) of their "universality", that is their "independence" on the fine details of the potentials. This paragraph will support once more this point of view; indeed we will see that these generalized "specific heats" do not really depend on the potentials but only on the spectral curve of the model and can be described by geometrical objects directly linked only to the curve itself. That is to say that these second order observables will be the same for any pairs of potentials for which the conformal structure of the spectral curve is isomorphic.

Of course this is no *proof* of universality, since there is no (rigorous, to our knowledge) proof that the free energy as defined in this paper is really obtained from some scaling limit of the partition function of the matrix model. Such a proof exists in the one-matrix model and uses rather sophisticated tools (Riemann Hilbert problem) [40]. A first step in this direction has already been taken by collaborators and the author in [13].

On the other hand we will see in section 3 that universality does not hold for third order observables, which will not depend purely on the conformal structure of the spectral curve but on the functions P and Q explicitly.

Let us start with $\partial_{u_K}\partial_{u_J}\mathcal{F}$

$$
2i\pi\partial_{u_K}\partial_{u_J}\mathcal{F} = \frac{1}{K}\partial_{u_J}\oint_{\infty_Q}PQ^K\mathrm{d}Q = \frac{1}{K}\oint_{\infty_Q}(\partial_{u_J}P)_Q Q^K\mathrm{d}Q\,. \tag{2.27}
$$

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Let us focus our attention on the differential $(\partial_{u_K} P)_Q \mathrm{d}Q$ (or $(\partial_{v_J} P)_Q \mathrm{d}Q$), where the subscript indicates that the corresponding quantity is kept fixed under differentiation. It follows from the definition of the coordinates on the moduli space and from eqs. (2.1) that

$$\begin{aligned}
(\partial_{u_K} P)_Q \mathrm{d}Q &= \left(Q^{K-1} + \mathcal{O}(Q^{-2})\right) \mathrm{d}Q \qquad \text{near } \infty_Q \\
(\partial_{u_K} P)_Q \mathrm{d}Q &= -(\partial_{u_K} Q)_P \mathrm{d}P = \mathcal{O}(P^{-2}) \mathrm{d}P \qquad \text{near } \infty_P \\
\oint_{a_j} (\partial_{u_K} P)_Q \mathrm{d}Q &= \partial_{u_K} \epsilon_j = 0 \,,
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
(\partial_{v_J} P)_Q \mathrm{d}Q &= -(\partial_{v_J} Q)_P \mathrm{d}P = \left(-P^{J-1} + \mathcal{O}(P^{-2})\right) \mathrm{d}P \qquad \text{near } \infty_P \\
(\partial_{v_J} Q)_P \mathrm{d}P &= \mathcal{O}(Q^{-2}) \mathrm{d}Q \qquad \text{near } \infty_Q \\
\oint_{a_j} (\partial_{v_J} P)_Q \mathrm{d}Q &= -\partial_{v_J} \epsilon_j = 0 \,.
\end{aligned} \tag{2.29}$$

In these formulas we have used repeatedly (and we will use it many times) the so-called *thermodynamic identity* (or *reciprocity*)

$$(\partial P)_Q \mathrm{d}Q = -(\partial Q)_P \mathrm{d}P \,, \tag{2.30}$$

where ∂ denotes any derivative. This is immediately obtained by differentiating P as a composite function with the (local) inverse of Q.

Note that the only singularities of these differentials are at the two marked points (see the discussion on the third-order observables for a proof that there are no poles at the branch-points). In other words $(\partial_{u_K} P)_Q \mathrm{d}Q$ and $(\partial_{v_J} P)_Q \mathrm{d}Q$ are abelian differentials of the second kind (i.e. with poles but no residues), normalized (i.e. the a-periods vanish) with a pole of degree $K+1$ or $J+1$ respectively only at ∞_Q or ∞_P respectively. It is immediate that this differential is uniquely determined and can be expressed in terms of the Bergman kernel as follows

$$\begin{aligned}
(\partial_{u_K} P)_Q \mathrm{d}Q(\zeta) &= \frac{1}{2i\pi\, K} \oint_{\infty_Q} Q^K(\widetilde{\zeta}) \Omega(\zeta, \widetilde{\zeta}) \,, \\
(\partial_{v_J} P)_Q \mathrm{d}Q(\zeta) &= -(\partial_{v_J} Q)_P \mathrm{d}P = -\frac{1}{2i\pi\, J} \oint_{\infty_P} P^J(\widetilde{\zeta}) \Omega(\zeta, \widetilde{\zeta}) \,.
\end{aligned} \tag{2.31}$$

Indeed, when $\zeta \sim \infty_Q$ we have (using $z = 1/Q(\zeta)$ as local coordinate)

$$\begin{aligned}
\frac{1}{2i\pi J} \oint_{\infty_Q} z^{-J} \Omega(\zeta, \widetilde{\zeta}) &= \frac{1}{2i\pi J} \oint_{\infty_Q} z^{-J} \left[\frac{1}{(z-z')^2} + \mathcal{O}(1)\right] \mathrm{d}z \mathrm{d}z' \\
&= \left(-z^{-J-1} + \mathcal{O}(1)\right) \mathrm{d}z = \left(Q^{J-1} + \mathcal{O}(Q^{-2})\right) \mathrm{d}Q \,.
\end{aligned} \tag{2.32}$$

It then follows that

$$\partial_{u_J} \partial_{u_K} \mathcal{F} = \frac{1}{(2i\pi)^2 KJ} \oint_{\infty_Q} \oint_{\infty_Q} Q^K(\zeta) Q^J(\widetilde{\zeta}) \Omega(\zeta, \widetilde{\zeta}) \,. \tag{2.33}$$

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By similar arguments one obtains also the formulas

$$
\begin{aligned}
\partial_{u_K}\partial_{v_J}\mathcal{F} &= \frac{1}{2i\pi K}\oint_{\infty_Q} Q^K(\partial_{v_J}P)_Q \mathrm{dQ} \overset{\text{thermodynamic identity}}{=} \\
&= -\frac{1}{2i\pi K}\oint_{\infty_Q} Q^K(\partial_{v_J}Q)_P \mathrm{d}P \\
&= \frac{-1}{(2i\pi)^2 KJ}\oint_{\infty_Q}\oint_{\infty_P} \Omega(\zeta,\widetilde{\zeta})Q^K(\zeta)P^J(\widetilde{\zeta}) \qquad (2.34) \\
\partial_{v_J}\partial_{v_K}\mathcal{F} &= \frac{1}{(2i\pi)^2 KJ}\oint_{\infty_P}\oint_{\infty_P} P^K(\zeta)P^J(\widetilde{\zeta})\Omega(\zeta,\widetilde{\zeta}) \,. \qquad (2.35)
\end{aligned}
$$

We now compute the second derivatives w.r.t. the filling fractions; to that extent we notice that $(\partial_{\epsilon_j}P)_Q\mathrm{d}Q$ is holomorphic everywhere because

$$
(\partial_{\epsilon_j}P)_Q\mathrm{d}Q = \begin{cases} \mathcal{O}(Q^{-2})\mathrm{d}Q & \text{near } \infty_Q \\ = -(\partial_{\epsilon_j}Q)_P\mathrm{d}P = \mathcal{O}(P^{-2})\mathrm{d}P & \text{near } \infty_P \,. \end{cases} \qquad (2.36)
$$

Thus it is regular at the marked points and has no other singularities (the poles of the variation at Q fixed of P cancel with the zeroes of $\mathrm{d}Q$; see discussion in section 3). Moreover it satisfies

$$
2i\pi\delta_{jk} = 2i\pi\partial_{\epsilon_j}\epsilon_k = \partial_{\epsilon_j}\oint_{a_k} P\mathrm{d}Q = \oint_{a_k}(\partial_{\epsilon_j}P)_Q\mathrm{d}Q \,. \qquad (2.37)
$$

Therefore $(\partial_{\epsilon_j}P)_Q\mathrm{d}Q = 2i\pi\omega_j$ where $\{\omega_j\}_{j=1\ldots g}$ is the basis of normalized abelian holomorphic differentials. As a consequence we obtain

$$
\partial_{\epsilon_j}\partial_{\epsilon_k}\mathcal{F} = \oint_{b_k}(\partial_{\epsilon_j}P)_Q\mathrm{d}Q = 2i\pi\oint_{b_k}\omega_j = \oint_{b_k}\oint_{b_j}\Omega = 2i\pi\,\tau_{jk}\,, \qquad (2.38)
$$

where $\tau = [\tau_{ij}]_{i,j=1\ldots g}$ is the period matrix for the holomorphic curve.

The other mixed derivatives are easily computed along the same lines as above to be

$$
\begin{aligned}
\partial_{u_K}\partial_{\epsilon_j}\mathcal{F} &= \frac{1}{2i\pi K}\oint_{\zeta\in b_j}\oint_{\infty_Q}\Omega(\zeta,\zeta')Q^K(\zeta') \\
\partial_t\partial_{u_K}\mathcal{F} &= \frac{-1}{2i\pi K}\oint_{\infty_Q} Q^K\mathrm{d}S \\
\partial_t\partial_{v_J}\mathcal{F} &= \frac{1}{2i\pi J}\oint_{\infty_P} P^J\mathrm{d}S \\
\partial_t\partial_{\epsilon_j}\mathcal{F} = \oint_{b_j}\mathrm{d}S &= \mathfrak{u}_j(\infty_Q) - \mathfrak{u}_j(\infty_P) \\
\partial_t^2\mathcal{F} &= \ln(\gamma\widetilde{\gamma})\,, \\
\ln(\gamma) &:= -\mathop{\mathrm{res}}_{\infty_Q}\ln\left(\frac{Q}{\lambda}\right)\mathrm{d}S\,, \qquad \ln(\widetilde{\gamma}) := \mathop{\mathrm{res}}_{\infty_P}\ln\left(P\lambda\right)\mathrm{d}S\,. \qquad (2.39)
\end{aligned}
$$

(Note that the product $\gamma\widetilde{\gamma}$ does not depend on the arbitrary multiplicative constant entering the definition of the multivalued function λ). The first two relations are obvious recalling that [1]

$$
\mathrm{d}S = -(\partial_t P)_Q\mathrm{d}Q = (\partial_t Q)_P\mathrm{d}P\,. \qquad (2.40)
$$

The third relation is a well known result for the third-kind abelian differential.

We shall prove the last relation: we start from the first derivative of $\mathcal{F}$ (μ, also called chemical potential) and its expression given in our previous paper [1]

$$
\begin{aligned}
2i\pi\mu &= 2i\pi\partial_t\mathcal{F} \\
&= \oint_{\infty_Q}\left(V_1(Q)-t\ln\left(\frac{Q}{\lambda}\right)\right)\mathrm{d}S-\oint_{\infty_P}\left(V_2(P)-t\ln(P\lambda)\right)\mathrm{d}S-\oint_{\infty_Q}PQ\mathrm{d}S+ \\
&\quad +2i\pi\sum_{j=1}^{g}\epsilon_j\oint_{b_j}\mathrm{d}S \\
&= \oint_{\infty_Q}\ln\left(\frac{Q}{\lambda}\right)P\mathrm{d}Q+\oint_{\infty_P}\ln\left(P\lambda\right)Q\mathrm{d}P-\oint_{\infty_Q}PQ\mathrm{d}S+2i\pi\sum_{j=1}^{g}\epsilon_j\oint_{b_j}\mathrm{d}S
\end{aligned}
\tag{2.41}
$$

We can therefore compute

$$
\begin{aligned}
2i\pi\partial_t\mu &= -\oint_{\infty_Q}\frac{1}{\lambda}\left(\partial_t\lambda\right)_Q P\mathrm{d}Q+\overbrace{\oint_{\infty_Q}\ln\left(\frac{Q}{\lambda}\right)\overbrace{(\partial_tP)_Q\mathrm{d}Q}^{=-\mathrm{d}S}}^{:=2i\pi\ln\gamma}+ \\
&\quad +\oint_{\infty_P}\frac{1}{\lambda}\left(\partial_t\lambda\right)_P Q\mathrm{d}P+\overbrace{\oint_{\infty_P}\ln\left(P\lambda\right)\overbrace{(\partial_tQ)_P\mathrm{d}P}^{=\mathrm{d}S}}^{=2i\pi\ln\widetilde{\gamma}}- \\
&\quad -\oint_{\infty_Q}P\,\overbrace{(\partial_tQ)_\lambda\frac{\mathrm{d}\lambda}{\lambda}}^{=-\frac{(\partial_t\lambda)_Q}{\lambda}\mathrm{d}Q}-\oint_{\infty_Q}Q\,\overbrace{(\partial_tP)_\lambda\frac{\mathrm{d}\lambda}{\lambda}}^{=-\frac{(\partial_t\lambda)_P}{\lambda}\mathrm{d}P}+2i\pi\sum_{j=1}^{g}\epsilon_j\oint_{b_j}(\partial_t\mathrm{d}S)_P \\
&= 2i\pi\ln(\gamma\widetilde{\gamma})\,,
\end{aligned}
\tag{2.42}
$$

where we have used Riemann bilinear identity to move the residue of the following term on line (2.42)

$$
\begin{aligned}
\oint_{\infty_P}\frac{1}{\lambda}\left(\partial_t\lambda\right)_P Q\mathrm{d}P &= -\oint_{\infty_P}\frac{1}{\lambda}\left(\partial_t\lambda\right)_P Q\mathrm{d}P+ \\
&\quad +\sum_{j=1}^{g}\overbrace{\oint_{a_j}Q\mathrm{d}P}^{=-2i\pi\epsilon_j}\left[\oint_{b_j}(\partial_t\mathrm{d}S)_P-\oint_{b_j}Q\mathrm{d}P\underbrace{\oint_{a_j}(\partial_t\mathrm{d}S)_P}_{=\partial_t\oint_{a_j}\mathrm{d}S=0}\right].
\end{aligned}
\tag{2.43}
$$

This concludes our analysis of the second order observable of the model. We repeat here for the sake of completeness that we are aware of a prior derivation in [28] and [26] for the case of conformal maps and genus 0 (one-cut case [25]).

2.3 Extension to formal power series for the potentials

The present derivation for the second derivatives does not rely on formal manipulations involving higher u_K or v_J. However it gives a deeper insight to use such a formalism. To that end we will now think of the two potentials V_1 and V_2 as infinite power series, without

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concern about their convergence. This is often done in the physical literature [19, 32] and yields the correct results in a faster and possibly more elegant way. To this end we define the *puncture* operators

$$\frac{\delta}{\delta V_1(q)} := \sum_{K=1}^{\infty} q^{-K-1} K \partial_{u_K}\,; \qquad \frac{\delta}{\delta V_2(p)} := \sum_{J=1}^{\infty} p^{-J-1} J \partial_{u_J}\,. \tag{2.44}$$

Using this formalism we realize that our free energy $\mathcal{F}_g$ is nothing but the *generating function* of the Bergman kernel on the selected curve

$$\frac{\delta^2 \mathcal{F}}{\delta V_1(q)\delta V_1(q')} = -\frac{1}{4\pi^2}\oint_{\infty_Q}\oint_{\infty_Q} \frac{\Omega(\zeta,\zeta')}{(q-Q(\zeta))(q'-Q(\zeta'))} = \frac{\Omega(\zeta(q),\zeta(q'))}{\mathrm{d}Q(\zeta(q))\mathrm{d}Q(\zeta(q'))}\,, \tag{2.45}$$

where the integrals are to be read in a formal sense of inverse power series in q, q', coefficient by coefficient; indeed each coefficient is precisely given by formula (2.33). In eq. (2.45) the notation $\zeta(q)$ means the point on the curve that projects on the *physical* sheet of the projection Q. The *physical* sheet is the sheet that has no cuts extending to infinity. On the physical sheet the formula is not formal; on the other sheets it should be properly understood as analytic continuation.

Similarly we have the other formulas where the P-projection is involved

$$\begin{aligned}
\frac{\delta^2 \mathcal{F}}{\delta V_2(p)\delta V_2(p')}\mathrm{d}p\,\mathrm{d}p' &= -\frac{1}{4\pi^2}\mathrm{d}p\,\mathrm{d}p'\oint_{\infty_P}\oint_{\infty_P} \frac{\Omega(\zeta,\zeta')}{(p-P(\zeta))(p'-P(\zeta'))} \\
&= \frac{\Omega(\widetilde{\zeta}(p),\widetilde{\zeta}(p'))}{\mathrm{d}P(\widetilde{\zeta}(p))\mathrm{d}P(\widetilde{\zeta}(p'))}\,, \\
\frac{\delta^2 \mathcal{F}}{\delta V_1(q)\delta V_2(p')}\mathrm{d}q\,\mathrm{d}p' &= -\frac{1}{4\pi^2}\mathrm{d}q\mathrm{d}p'\oint_{\infty_Q}\oint_{\infty_P} \frac{\Omega(\zeta,\zeta')}{(q-Q(\zeta))(p'-P(\zeta'))} \\
&= \frac{\Omega(\zeta(q),\zeta(p'))}{\mathrm{d}Q(\zeta(q))\mathrm{d}P(\widetilde{\zeta}(p'))}\,.
\end{aligned} \tag{2.46}$$

To be more precise, these formulas are exact (modulo the issue of convergence of the potentials) only in a neighborhood of the respective punctures where Q and P provide univalued coordinates.

It is interesting to recast the problem upside-down: let it be given a smooth curve of genus g Σ_g with Bergman kernel Ω (which implies a fixation of symplectic basis in the homology of the curve) and two marked points $\infty_{Q,P}$ such that:

- there exist two functions Q, P with simple poles at the respective marked points;

- the only other singularity for the functions above are essential singularities at the other (respective) marked point.

In this setting we can define "coordinates" u_K, v_J by means of formulas (2.3) and then the expansion of the Bergman kernel around the punctures in the coordinates provided by the functions P, Q are the second derivatives of a "tau" function.

2.3.1 Relation with conformal maps

The reader trying to bridge the present setting with [31] should be warned that the puncture operator here is *not* the "infinitesimal deformation" operator ibidem. Their operator would be written in our notation

$$\frac{\delta}{\delta V_1'(x)} = \sum_{K=1}^{\infty} x^{-K} \partial_{u_K} \,, \qquad \frac{\delta}{\delta V_2'(y)} = \sum_{J=1}^{\infty} y^{-J} \partial_{v_J} \,, \tag{2.47}$$

with the relation (e.g. for the $x-x$ two-puncture case)

$$\frac{\mathrm{d}}{\mathrm{d}x_1} \frac{\mathrm{d}}{\mathrm{d}x_2} \frac{\delta}{\delta V_1'(x_1)} \frac{\delta}{\delta V_1'(x_2)} \mathcal{F}_g = \frac{\delta}{\delta V_1(x_1)} \frac{\delta}{\delta V_1(x_2)} \mathcal{F}_g \,. \tag{2.48}$$

Therefore their double-deformation is the antiderivative of our double puncture. This is why in [31] they have the logarithm of the prime form[6] whereas we have the Bergman kernel. Since in this paper we will not investigate Hirota equations (i.e. Fay's identities) which involve the prime form more naturally than the Bergman kernel, we will not pursue this observation any further.

3. Third derivatives: Beltrami differentials

In this section we compute the third derivatives in terms of the canonical structures and data of our moduli space. We must note here that in the case of the tau-function for conformal maps of connected domains (i.e. genus 0 case) constructed by Marshakov, Krichever, Wiegmann, Zabrodin et. al. these formulas are known [32] (see appendix B). In order to obtain their "residue formulas" from the formulas we are going to write down, we should restrict ourselves to the genus zero case and impose that $P(\lambda) = \overline{Q}(1/\overline{\lambda})$, λ being the uniformizing parameter of the rational curve with $\lambda(\infty_Q) = \infty$ and $\lambda(\infty_P) = 0$ (see appendix B). However our formulas cover a more general case that would includes the situation of conformal maps of multiply-connected domains [31], were we to impose some restriction of reality on P and Q (as explained in section 2)

When computing a variation in the moduli of our problem $\{u_K, v_J, t, \epsilon_j\}$, we introduce a deformation of the conformal structure of the curve. We will see that this deformation is equivalent to introducing simple poles at the branch-points of one or the other of the two projections $Q : \Sigma_g \to \mathbb{C}P^1$ and $P : \Sigma_g \to \mathbb{C}P^1$, depending on which among Q and P we keep fixed under the variation. Let us denote by $q_\mu, \mu = 1, \ldots, d_2 + 1 + 2g$ and p_ν, $\nu = 1, \ldots, d_1 + 1 + 2g$ such points and then by Q_μ, P_ν the corresponding critical values; here we are implicitly making an assumption of genericity of the potentials, in that we will consider these branch-points as simple. Were we to consider more degenerate cases (*critical potentials*) we would have to modify some of the formulas below (mainly the Beltrami differentials). Note that critical potentials (i.e. potentials which are fine-tuned so as to have degenerate singularities of the maps P and Q) are very important for applications to conformal field theories in that they provide instances of minimal conformal models.

[6]Here we are speaking rather loosely; we should also fix a coordinate and consider the prime form as a bi-function by trivializing the spinor bundle.

Coming back to our generic situation, let us consider the case of the Q-projection, the other case being totally similar. The projection $Q : \Sigma_g \to \mathbb{C}P^1$ determines the conformal structure of the curve and hence it suffices to consider the variation of the structure arising from the deformation of this projection. Consider first the branch-points of the Q-projection and introduce the local parameter in a neighborhood of the critical point q_μ

$$\zeta_\mu = \sqrt{Q - Q_\mu} \,. \tag{3.1}$$

As we know, the critical values of both projections Q and P are *not* independent coordinates in our moduli space; they vary when performing a variation of our moduli. Let us call ∂ any infinitesimal variation of our moduli. Under such a variation Q_μ also undergoes some variation ∂Q_μ. Suppose now we have a germ of function F in a neighborhood of q_μ (which may also depend explicitly on the moduli) and we make the variation ∂ at Q-value fixed. Then

$$(\partial F(\zeta_\mu))_Q = -\frac{\partial Q_\mu}{2\zeta_\mu}\frac{\mathrm{d}F(\zeta_\mu)}{\mathrm{d}\zeta_\mu} + \partial_\partial F(\zeta_\mu) \,, \tag{3.2}$$

where $\partial_\partial F$ denotes the variation of F coming from the explicit dependence on the moduli. We see immediately from (3.2) that, generically, the variation has a simple pole at each branch-point of the Q-projection. Ditto for the branch-points of P when computing a variation at P-value fixed.

A variation of the moduli induces a variation in the conformal structure of the underlying real surface Σ_g; these infinitesimal variations are described by the so-called "Beltrami differentials". They consist of differentials of the form

$$\mu(z,\overline{z})\frac{\mathrm{d}\overline{z}}{\mathrm{d}z} \ \in L^\infty(\Sigma_g) \ , \tag{3.3}$$

and enter into Rauch's formula [41] for the variation of the Bergman kernel

$$\delta_\mu \Omega(\zeta,\zeta') = \int\int_{\Sigma_g} \mu(\zeta'')\Omega(\zeta'',\zeta)\Omega(\zeta'',\zeta') \,. \tag{3.4}$$

It is shown in many places [39] and [41]–[43] that varying the image of a branch-point b of a covering $\varphi : \Sigma_g \to \mathbb{C}P^1$ corresponds to the following Beltrami differential (in terms of a local coordinate z centered at b)

$$\mu_{Sc} = -\frac{1}{2\epsilon^2}\mathbb{1}_\epsilon(|z|)\frac{\mathrm{d}\overline{z}}{\mathrm{d}z} \,, \tag{3.5}$$

where $\mathbb{1}_\epsilon$ is the characteristic function of the ϵ disc and ϵ is a parameter small enough so that no other branch-points fall within the disc. This is called "Schiffer" variation and corresponds to varying the critical value of a simple branch-point; for higher order branch-point see [39]. Plugging this into (3.4) and using Green's theorem followed by Cauchy's residues theorem one obtains simply a residue

$$\delta_{\mu_{Sc}}\Omega(\zeta,\zeta') = \operatorname*{res}_b \frac{\Omega(\zeta,\zeta'')\Omega(\zeta',\zeta'')}{\mathrm{d}\varphi(\zeta'')} \,. \tag{3.6}$$

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Our setting is slightly different, in that we have *two* projections P and Q and the respective branching values are not independent moduli but vary together with the $\{u_K, v_J, \epsilon_j, t\}$ moduli of our problem. Therefore all we need in order to find the Beltrami differentials corresponding to these variations is to be able to find the coefficients ∂Q_μ and ∂P_ν that appear in eq. (3.2). To this end let us now consider the case of a differential of the form $F\mathrm{d}Q$. It follows from the above discussion (and we have already used this fact implicitly to compute the first variations of $P\mathrm{d}Q$) that its variation at Q fixed will be holomorphic at the branch-points because the differential $\mathrm{d}Q$ has simple zeroes there which cancel the poles of the variation of F.

Suppose now that for some reason we have a way of identifying independently what differential is $\omega := (\partial F)_Q \mathrm{d}Q$ Then we could compute the desired coefficients ∂Q_μ as in the following formula

$$\begin{aligned} \frac{\omega}{\mathrm{d}\zeta_\mu}(q_\mu) &= -\partial Q_\mu \frac{\mathrm{d}F}{\mathrm{d}\zeta_\mu}(q_\mu) \quad \text{that is...} \\ \partial Q_\mu &= -\frac{\omega}{\mathrm{d}F}(q_\mu)\,. \end{aligned} \tag{3.7}$$

We now apply the above to the case $F = P$ and the differential $P\mathrm{d}Q$; in this case we have already discovered that

1. the derivative of $P\mathrm{d}Q$ w.r.t. ϵ_j is the holomorphic normalized abelian differential $(\partial_{\epsilon_j} P)_Q \mathrm{d}Q = \omega_j$;

2. the derivatives of $P\mathrm{d}Q$ w.r.t. u_K or v_J are given by eq. (2.31);

3. the derivative of $P\mathrm{d}Q$ w.r.t. t is the normalized abelian integral $(\partial_t P)_Q \mathrm{d}Q = \mathrm{d}S$ of the third kind with poles at the marked points.

Therefore the parameter ∂Q_μ and ∂P_ν are given by the following formulas

$$\begin{aligned} \partial_{\epsilon_j} Q_\mu &= -\left.\frac{\omega_j}{\mathrm{d}P}\right|_{q_\mu} = -\frac{1}{2i\pi}\left.\frac{\oint_{\widetilde{\zeta}\in b_j} \Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}P(\zeta)}\right|_{\zeta=q_\mu} \\ \partial_{u_K} Q_\mu &= -\frac{1}{2i\pi K}\left.\frac{\oint_{\infty_Q} Q(\widetilde{\zeta})^K \Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}P(\zeta)}\right|_{\zeta=q_\mu} \\ \partial_{v_J} Q_\mu &= \frac{1}{2i\pi J}\left.\frac{\oint_{\infty_P} P(\widetilde{\zeta})^J \Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}P(\zeta)}\right|_{\zeta=q_\mu} \\ \partial_t Q_\mu &= -\left.\frac{\int_{\infty_Q}^{\infty_P} \Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}P(\zeta)}\right|_{\zeta=q_\mu}. \end{aligned} \tag{3.8}$$

Repeating all of the above but interchanging the rôle of P and Q we find (note the opposite signs, coming from the thermodynamic identity)

$$\begin{aligned}
\partial_{\epsilon_j} P_\nu &= \left.\frac{\omega_j}{\mathrm{d}Q}\right|_{p_\nu} = \frac{1}{2i\pi}\left.\frac{\oint_{\widetilde{\zeta}\in b_j}\Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}Q(\zeta)}\right|_{\zeta=p_\nu} \\
\partial_{u_K} P_\nu &= \frac{1}{2i\pi K}\left.\frac{\oint_{\infty_P} Q(\widetilde{\zeta})^K\Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}Q(\zeta)}\right|_{\zeta=p_\nu} \\
\partial_{v_J} P_\nu &= -\frac{1}{2i\pi J}\left.\frac{\oint_{\infty_P} P(\widetilde{\zeta})^J\Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}Q(\zeta)}\right|_{\zeta=p_\nu} \\
\partial_t P_\nu &= \left.\frac{\int_{\infty_Q}^{\infty_P}\Omega(\zeta,\widetilde{\zeta})}{\mathrm{d}Q(\zeta)}\right|_{\zeta=p_\nu}.
\end{aligned} \tag{3.9}$$

In general we can write the formula (∂ denoting an arbitrary variation)

$$\begin{aligned}
\partial Q_\mu &= -\left.\frac{(\partial P)_Q \mathrm{d}Q}{\mathrm{d}P}\right|_{q_\mu} = (\partial Q)_P(q_\mu) \\
\partial P_\nu &= -\left.\frac{(\partial Q)_P \mathrm{d}P}{\mathrm{d}Q}\right|_{p_\nu} = (\partial P)_Q(p_\nu)\,.
\end{aligned} \tag{3.10}$$

We are now in a position of expressing the Beltrami differentials corresponding to a variation in our moduli as in

$$\mu_{(\partial)} = \sum_{\mu=1}^{d_2+1+2g}\partial Q_\mu \mu_{Sc}^{(q_\mu)} \equiv \sum_{\nu=1}^{d_1+1+2g}\partial P_\nu \mu_{Sc}^{(p_\nu)}\,, \tag{3.11}$$

where the notation $\mu_{Sc}^{(\zeta)}$ denotes the Schiffer variation centered at the point ζ. The equivalence sign $\equiv$ means that the two expressions determine the same variation of conformal structure. Indeed we should remind that any two Beltrami differentials μ and $\widetilde{\mu}$ determine the same variation of the conformal structure if for any quadratic holomorphic differential Φ we have

$$\iint_{\Sigma_g}\mu\Phi = \iint_{\Sigma_g}\widetilde{\mu}\,\Phi\,. \tag{3.12}$$

Indeed, for $\Phi\in H^0(\mathcal{K}^2)$ a holomorphic quadratic differential we have

$$\begin{aligned}
\sum_{\mu=1}^{d_2+1+2g}\partial Q_\mu\iint_{\Sigma_g}\mu_{Sc}^{(q_\mu)}\Phi &= \sum_{\mu=1}^{d_2+1+g}\partial Q_\mu \mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{\Phi}{\mathrm{d}Q} = -\sum_{\mu=1}^{d_2+1+g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{(\partial P)_Q\mathrm{d}Q\,\Phi}{\mathrm{d}Q\mathrm{d}P} \overset{(\star)}{=} \\
&= \sum_{\nu=1}^{d_1+1+g}\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{(\partial P)_Q\mathrm{d}Q\,\Phi}{\mathrm{d}Q\mathrm{d}P} = \sum_{\nu=1}^{d_1+1+g}\partial P_\nu\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{\Phi}{\mathrm{d}P} = \\
&= \sum_{\nu=1}^{d_1+1+2g}\partial P_\nu\iint_{\Sigma_g}\mu_{Sc}^{(p_\nu)}\Phi\;,
\end{aligned} \tag{3.13}$$

thus proving the equivalence of the two Beltrami differentials. The equality marked ($\star$) follows from the fact that the differential $\frac{(\partial P)_Q \mathrm{d}Q\,\Phi}{\mathrm{d}Q\mathrm{d}P}$ has poles only at the branch-points of P and Q, and the sum of all residues must vanish.[7]

From Rauch variational formula (3.4) (see also [39]) follows then that the variation of the Bergman kernel under such a Beltrami differential is

$$\begin{aligned}
(\partial\Omega(\zeta,\zeta'))_Q &= \sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \partial Q_\mu \frac{\Omega(\zeta,\widetilde{\zeta})\Omega(\zeta',\widetilde{\zeta})}{\mathrm{d}Q(\widetilde{\zeta})} \\
&= \sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \frac{(\partial P)_Q \mathrm{d}Q(\widetilde{\zeta})\Omega(\zeta,\widetilde{\zeta})\Omega(\zeta',\widetilde{\zeta})}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})} \qquad (3.14)\\
(\partial\Omega(\zeta,\zeta'))_P &= \sum_{\nu=1}^{d_1+1+2g} \operatorname*{res}_{\widetilde{\zeta}=p_\nu} \partial P_\nu \frac{\Omega(\zeta,\widetilde{\zeta})\Omega(\zeta',\widetilde{\zeta})}{\mathrm{d}P(\widetilde{\zeta})} \\
&= \sum_{\nu=1}^{d_1+1+2g} \operatorname*{res}_{\widetilde{\zeta}=p_\nu} \frac{(\partial Q)_P \mathrm{d}P(\widetilde{\zeta})\Omega(\zeta,\widetilde{\zeta})\Omega(\zeta',\widetilde{\zeta})}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})}\,. \qquad (3.15)
\end{aligned}$$

Using the expressions above for the coefficients ∂Q_μ and ∂P_ν we obtain then

$$\begin{aligned}
(\partial_{\epsilon_j}\Omega(\zeta,\zeta'))_Q &= -\frac{1}{2i\pi} \sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \frac{\oint_{\zeta''\in b_j} \Omega(\zeta,\widetilde{\zeta})\Omega(\widetilde{\zeta},\zeta'')\Omega(\widetilde{\zeta},\zeta')}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})} \\
(\partial_{u_K}\Omega(\zeta,\zeta'))_Q &= -\frac{1}{2i\pi K} \sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \frac{\oint_{\zeta''\sim\infty_Q} Q(\zeta'')^K \Omega(\zeta,\widetilde{\zeta})\Omega(\widetilde{\zeta},\zeta'')\Omega(\widetilde{\zeta},\zeta')}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})} \\
(\partial_{v_J}\Omega(\zeta,\zeta'))_Q &= \frac{1}{2i\pi J} \sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \frac{\oint_{\zeta''\sim\infty_P} P(\zeta'')^J \Omega(\zeta,\widetilde{\zeta})\Omega(\widetilde{\zeta},\zeta'')\Omega(\widetilde{\zeta},\zeta')}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})} \\
(\partial_t\Omega(\zeta,\zeta'))_Q &= -\sum_{\mu=1}^{d_2+1+2g} \operatorname*{res}_{\widetilde{\zeta}=q_\mu} \frac{\int_{\zeta''=\infty_Q}^{\infty_P} \Omega(\zeta,\widetilde{\zeta})\Omega(\widetilde{\zeta},\zeta'')\Omega(\widetilde{\zeta},\zeta')}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})}\,. \qquad (3.16)
\end{aligned}$$

The variations at P fixed would be given by the same formulas as above but replacing the sum over the critical point of Q by the sum over the critical points of P and changing the overall sign.

The reason why we distinguish between the variations at Q or P fixed will be clearer in a moment.

Indeed let us compute now the third derivatives of $\mathcal{F}_g$. If for example we want to compute $\partial_{u_J}\partial_{u_L}\partial_{v_K}\mathcal{F}_g$ the simplest way is to leave the variation of v_K last so that we will

[7] The differentials $(\partial P)_Q \mathrm{d}Q$ have poles at most of degree d_2+2 at ∞_Q or d_1+2 at ∞_P but those are canceled by the poles of $\mathrm{d}P\mathrm{d}Q$ at those points, of degree d_2+3 and d_1+3 respectively.

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be varying at Q fixed;

$$\begin{aligned}
\partial_{u_J}\partial_{u_L}\partial_{v_K}\mathcal{F}_g &= \partial_{v_K}\partial_{u_J}\partial_{u_L}\mathcal{F}_g \\
&= \frac{1}{2i\pi}\partial_{v_K}\oint_{\infty_Q}\frac{Q^K}{K}(\partial_{u_L}P)_Q \mathrm{d}Q \\
&= \frac{1}{(2i\pi)^2 JL}\partial_{v_K}\oint_{\infty_Q}\oint_{\infty_Q} Q(\zeta)^J Q(\zeta')^L \Omega(\zeta,\zeta') \\
&= -\frac{1}{4\pi^2 JL}\oint_{\infty_Q}\oint_{\infty_Q} Q(\zeta)^J Q(\zeta')^L (\partial_{v_K}\Omega(\zeta,\zeta'))_Q \\
&= -\frac{1}{8i\pi^3 JLK}\times \\
&\quad\times \sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\widetilde{\zeta}=q_\mu}\oint_{\infty_Q}\oint_{\infty_Q}\oint_{\infty_P} Q(\zeta)^J Q(\zeta')^L P(\zeta'')^K \frac{\Omega(\zeta,\widetilde{\zeta})\Omega(\zeta',\widetilde{\zeta})\Omega(\zeta'',\widetilde{\zeta})}{\mathrm{d}P(\widetilde{\zeta})\mathrm{d}Q(\widetilde{\zeta})} \\
&= \sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\widetilde{\zeta}=q_\mu}\frac{(\partial_{v_K}P)_Q\mathrm{d}Q(\partial_{u_J}P)_Q\mathrm{d}Q(\partial_{u_L}P)_Q\mathrm{d}Q}{\mathrm{d}Q\mathrm{d}P}\,.
\end{aligned} \tag{3.17}$$

Performing the variation at P fixed, since P is not a local coordinate near the marked point ∞_Q, would introduce unnecessary complication to the formula (basically adding an extra residue to it). Thus we need to pay the necessary care in choosing which among P and Q needs to be kept fixed in order to have the simplest formulas.

With this *caveat* in mind can thus now compute the third derivatives of the free energy. The resulting formulas should be compared with the residue formulas in [32, 26] (which would correspond to genus $g = 0$ here) The formulas are simplified if we introduce the quadri-differential

$$\Omega^{(3,1)}(\zeta_1,\zeta_2,\zeta_3;\zeta) = \frac{\Omega(\zeta_1,\zeta)\Omega(\zeta_2,\zeta)\Omega(\zeta_3,\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\,, \tag{3.18}$$

and the notations

$$\begin{aligned}
\mathcal{U}_K(\zeta) &:= (\partial_{u_K}P)_Q\mathrm{d}Q(\zeta) = \frac{1}{2i\pi K}\oint_{\infty_Q} Q(\zeta')^K\Omega(\zeta,\zeta')\,, \\
\mathcal{V}_J(\zeta) &:= (\partial_{v_J}P)_Q\mathrm{d}Q(\zeta) = -\frac{1}{2i\pi J}\oint_{\infty_P} P(\zeta')^J\Omega(\zeta,\zeta')\,.
\end{aligned} \tag{3.19}$$

We also recall that

$$\omega_j = (\partial_{\epsilon_j}P)_Q\mathrm{d}Q\,, \qquad \mathrm{d}S = -(\partial_t P)_Q\mathrm{d}Q\,, \tag{3.20}$$

are the normalized abelian differentials of the first and third kind respectively. With these definitions and reminders we have

$$\begin{aligned}
\partial_{\epsilon_i}\partial_{\epsilon_j}\partial_{\epsilon_k}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{\omega_i(\zeta)\omega_j(\zeta)\omega_k(\zeta)}{\mathrm{d}P\mathrm{d}Q} \\
&= -\frac{1}{(2i\pi)^3}\sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\oint_{\zeta_1\in b_i}\oint_{\zeta_2\in b_j}\oint_{\zeta_3\in b_k}\Omega^{(3,1)}(\zeta_1,\zeta_2,\zeta_3;\zeta) \\
&= \sum_{\nu=1}^{d_1+1+2g}\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{\omega_i(\zeta)\omega_j(\zeta)\omega_k(\zeta)}{\mathrm{d}P\mathrm{d}Q} = 2i\pi\partial_{\epsilon_i}\tau_{jk}
\end{aligned}$$

$$
\begin{aligned}
\partial_{\epsilon_i}\partial_{\epsilon_j}\partial_t\mathcal{F}_g &= -\frac{1}{(2i\pi)^2}\sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\oint_{\zeta_1\in b_i}\oint_{\zeta_2\in b_j}\int_{\zeta_3=\infty_Q}^{\infty_P}\Omega^{(3,1)}(\zeta_1,\zeta_2,\zeta_3;\zeta)\\
&= \sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{\omega_i(\zeta)\omega_j(\zeta)\mathrm{d}S(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = \sum_{\nu=1}^{d_1+1+2g}\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{\omega_i(\zeta)\omega_j(\zeta)\mathrm{d}S(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{\epsilon_i}\partial_t^2\mathcal{F}_g &= -\frac{1}{2i\pi}\sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\oint_{\zeta_1\in b_i}\int_{\zeta_2=\infty_Q}^{\infty_P}\int_{\zeta_3=\infty_Q}^{\infty_P}\Omega^{(3,1)}(\zeta_1,\zeta_2,\zeta_3;\zeta)\\
&= -\sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{\omega_i(\zeta)\mathrm{d}S(\zeta)^2}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = \sum_{\nu=1}^{d_1+1+2g}\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{\omega_i(\zeta)\mathrm{d}S(\zeta)^2}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\,.
\end{aligned}
\tag{3.21}
$$

The third derivative w.r.t. t has to be treated with some care, although the result is a formula of the same sort. For the purpose of this computation it is best to choose as basepoint defining λ the marked point ∞_Q, regularizing appropriately the integral

$$
\ln\lambda = \fint_{\infty_Q}\mathrm{d}S := \lim_{\zeta\to\infty_Q}\int_\zeta \mathrm{d}S-\ln(Q(\zeta)) = \lim_{\zeta\to\infty_Q}\int_\zeta \mathrm{d}S-\frac{1}{d_1}\ln(P(\zeta))+\frac{1}{d_1}\ln(v_{d_2+1})\,. \tag{3.22}
$$

This is just a convenient normalization of the function λ with the property that

$$
(\partial_t\ln(\lambda))_P = \int_{\infty_Q}(\partial_t\mathrm{d}S)_P\,,\qquad (\partial_t\ln(\lambda))_Q = \int_{\infty_Q}(\partial_t\mathrm{d}S)_Q\,. \tag{3.23}
$$

Using these properties we find

$$
\begin{aligned}
\partial_t^3\mathcal{F}_g &= \partial_t\ln(\gamma\tilde{\gamma}) = \partial_t\left(\mathop{\mathrm{res}}_{\infty_Q}\ln\left(\frac{Q}{\lambda}\right)\frac{\mathrm{d}\lambda}{\lambda}-\mathop{\mathrm{res}}_{\infty_P}\ln(P\lambda)\frac{\mathrm{d}\lambda}{\lambda}\right)\\
&= \mathop{\mathrm{res}}_{\infty_Q}\frac{(\partial_t Q)_\lambda\mathrm{d}\lambda}{Q\lambda}-\mathop{\mathrm{res}}_{\infty_P}\frac{(\partial_t P)_\lambda\mathrm{d}\lambda}{P\lambda}\\
&= -\mathop{\mathrm{res}}_{\infty_Q}\frac{(\partial_t\lambda)_Q\mathrm{d}Q}{Q\lambda}+\mathop{\mathrm{res}}_{\infty_P}\frac{(\partial_t\lambda)_P\mathrm{d}P}{P\lambda}\\
&= -\mathop{\mathrm{res}}_{\infty_Q}\frac{(\partial_t\ln(\lambda))_Q\mathrm{d}Q}{Q}+\mathop{\mathrm{res}}_{\infty_P}\frac{(\partial_t\ln(\lambda))_P\mathrm{d}P}{P}\\
&= -\mathop{\mathrm{res}}_{\infty_Q}\frac{\mathrm{d}Q\int_{\infty_Q}\int_{\infty_Q}^{\infty_P}(\partial_t\Omega)_Q}{Q}+\mathop{\mathrm{res}}_{\infty_P}\frac{\mathrm{d}P\int_{\infty_Q}\int_{\infty_Q}^{\infty_P}(\partial_t\Omega)_P}{P}\\
&= \overbrace{\int_{\infty_Q}^{\infty_Q}\int_{\infty_P}^{\infty_Q}(\partial_t\Omega)_Q}^{=0}-\int_{\infty_Q}^{\infty_P}\int_{\infty_Q}^{\infty_P}(\partial_t\Omega)_P\\
&= \sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\int_{\zeta_1=\infty_Q}^{\infty_P}\int_{\zeta_2=\infty_Q}^{\infty_P}\int_{\zeta_3=\infty_Q}^{\infty_P}\Omega^{(3,1)}(\zeta_1,\zeta_2,\zeta_3;\zeta)\\
&= \sum_{\mu=1}^{d_2+1+2g}\mathop{\mathrm{res}}_{\zeta=q_\mu}\frac{\mathrm{d}S(\zeta)^3}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = -\sum_{\nu=1}^{d_1+1+2g}\mathop{\mathrm{res}}_{\zeta=p_\nu}\frac{\mathrm{d}S(\zeta)^3}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\,.
\end{aligned}
\tag{3.24}
$$

The other derivatives are

$$
\begin{aligned}
\partial_{\epsilon_i}\partial_{\epsilon_j}\partial_{u_K}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\omega_j(\zeta)\omega_k(\zeta)\mathcal{U}_K(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\omega_j(\zeta)\omega_k(\zeta)\mathcal{U}_K(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{\epsilon_i}\partial_{\epsilon_j}\partial_{v_J}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\omega_j(\zeta)\omega_k(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\omega_j(\zeta)\omega_k(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{\epsilon_i}\partial_{u_K}\partial_{v_J}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\omega_j(\zeta)\mathcal{U}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\omega_j(\zeta)\mathcal{U}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_t\partial_{u_K}\partial_{v_J}\mathcal{F}_g &= \sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\mathrm{d}S(\zeta)\mathcal{U}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)} = -\sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\mathrm{d}S(\zeta)\mathcal{U}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\,.
\end{aligned}
\tag{3.25}
$$

In all of the above formulas we can move the sum over the residues at the critical points of either maps P and Q because the differential has poles only at those points and not at the marked points. Indeed the denominator has a pole of order d_1+3 and d_2+3 at ∞_Q and ∞_P respectively, and the pole of the numerator never exceeds these values. In the remaining derivatives, instead, the simplest form is obtained by summing over only the critical points of one of the two maps, case by case:

$$
\begin{aligned}
\partial_t^2\partial_{u_K}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\mathrm{d}S(\zeta)^2\mathcal{U}_K(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_t^2\partial_{v_J}\mathcal{F}_g &= \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\mathrm{d}S(\zeta)^2\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_t\partial_{u_K}\partial_{u_J}\mathcal{F}_g &= \sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\mathrm{d}S(\zeta)\mathcal{U}_K(\zeta)\mathcal{U}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{\epsilon_i}\partial_{u_K}\partial_{u_J}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\omega_i(\zeta)\mathcal{U}_K(\zeta)\mathcal{U}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{u_L}\partial_{u_K}\partial_{u_J}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\mathcal{U}_L(\zeta)\mathcal{U}_K(\zeta)\mathcal{U}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{v_L}\partial_{u_K}\partial_{u_J}\mathcal{F}_g &= -\sum_{\mu=1}^{d_2+1+2g}\operatorname*{res}_{\zeta=q_\mu}\frac{\mathcal{V}_L(\zeta)\mathcal{U}_K(\zeta)\mathcal{U}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_t\partial_{v_K}\partial_{v_J}\mathcal{F}_g &= -\sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\mathrm{d}S(\zeta)\mathcal{V}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{\epsilon_i}\partial_{v_K}\partial_{v_J}\mathcal{F}_g &= \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\omega_i(\zeta)\mathcal{V}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{v_L}\partial_{v_K}\partial_{v_J}\mathcal{F}_g &= \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\mathcal{V}_L(\zeta)\mathcal{V}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\\
\partial_{u_L}\partial_{v_K}\partial_{v_J}\mathcal{F}_g &= \sum_{\nu=1}^{d_1+1+2g}\operatorname*{res}_{\zeta=p_\nu}\frac{\mathcal{U}_L(\zeta)\mathcal{V}_K(\zeta)\mathcal{V}_J(\zeta)}{\mathrm{d}P(\zeta)\mathrm{d}Q(\zeta)}\,.
\end{aligned}
\tag{3.26}
$$

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Although we have not written it each time, it is clear that any of these derivatives is obtained by means of suitable integrals of the kernel $\Omega^{(3,1)}(\bullet,\bullet,\bullet;\zeta)$ over

1. a b-cycle corresponding to each variation w.r.t. ϵ's (times $\frac{1}{2i\pi}$);
2. a circle around ∞_Q against $Q^K/(2i\pi\, K)$ for each variation w.r.t. u_K's;
3. a circle around ∞_P against $-P^J/(2i\pi\, J)$ for each variation w.r.t. v_J's;
4. a path from ∞_P to ∞_Q for each variation w.r.t. t;

followed by minus the sum over the residues at the points $\zeta = q_\mu$ if there are two or more derivatives w.r.t the u_K's or two derivatives w.r.t. t and one w.r.t. u_K, or the sum of residues at the points $\zeta = p_\nu$'s if there are two or more derivatives w.r.t. the v_J's or two derivatives w.r.t. t and one w.r.t. v_J. In all other cases the choice of residues is immaterial.

Using the above results somewhat liberally in the case of potentials given by power series we obtain

Corollary 2 (Three-loops correlators) *Denoting by $\zeta(x)$ the inverse function of $Q(\zeta) = x$ and $\widetilde{\zeta}(y)$ the inverse function of $P(\zeta) = y$ on the respective physical sheets we can write the three-loop correlators as follows*

$$\left\langle \mathrm{Tr}\,\frac{1}{x-M_1}\,\mathrm{Tr}\,\frac{1}{x'-M_1}\,\mathrm{Tr}\,\frac{1}{x''-M_1}\right\rangle_{\mathrm{conn}} = -\frac{\sum_\mu \operatorname*{res}_{\zeta=q_\mu} \Omega^{(3,1)}(\zeta(x),\zeta(x'),\zeta(x'');\zeta)}{\mathrm{d}Q(\zeta(x))\mathrm{d}Q(\zeta(x'))\mathrm{d}Q(\zeta(x''))}$$

$$\left\langle \mathrm{Tr}\,\frac{1}{x-M_1}\,\mathrm{Tr}\,\frac{1}{x'-M_1}\,\mathrm{Tr}\,\frac{1}{y-M_2}\right\rangle_{\mathrm{conn}} = -\frac{\sum_\mu \operatorname*{res}_{\zeta=q_\mu} \Omega^{(3,1)}(\zeta(x),\zeta(x'),\widetilde{\zeta}(y);\zeta)}{\mathrm{d}Q(\zeta(x))\mathrm{d}Q(\zeta(x'))\mathrm{d}P(\widetilde{\zeta}(x''))}\,, \tag{3.27}$$

and similarly for the other two three-loop correlators but summing over p_ν's and with an overall minus sign.

Proof. The quotes are because a complete proof should address also the issue of convergence of the series involved. Indeed we can obtain the above correlators by applying the relevant puncture operators, which can be done only by considering infinite series for the potentials. At the end of this process we should set all the coefficients but a finite number to zero so that the final formula involves only a finite sum of residues. Since we are not addressing the summability of the series involved in this procedure we cannot claim rigor for this proof. With this in mind we nevertheless proceed on a formal level. For instance the first correlator would be

$$\begin{aligned}
&\sum_{K,K',K''=1}^{\infty} x^{-K-1}x'^{-K'-1}x''^{-K''-1}KK'K''\partial_{u_K}\partial_{u_{K'}}\partial_{u_{K''}}\mathcal{F}_g = \\
&= \frac{-1}{(2i\pi)^3}\sum_{K,K',K''=1}^{\infty}\oint_{\infty_Q}\oint_{\infty_Q}\oint_{\infty_Q}\sum_\mu \operatorname*{res}_{\zeta=q_\mu}\frac{Q(\zeta)^K}{x^K}\frac{Q(\zeta')^{K'}}{(x')^{K'+1}}\frac{Q(\zeta'')^{K''}}{(x'')^{K''+1}}\Omega^{(3,1)}(\zeta,\zeta',\zeta'';\zeta) \\
&= (\star)\,.
\end{aligned} \tag{3.28}$$

We can interchange the sums with the integrals (and sum the resulting geometric series) if the loops around ∞_Q project — e.g.— to circles in the Q-plane which leave x, x', x'' in the unbounded region, that is, if $|Q(\zeta)/x|$, etc. remain bounded and less than one. In this case we have then

$$(\star) = \frac{-1}{(2i\pi)^3}\oint_{\infty_Q}\oint_{\infty_Q}\oint_{\infty_Q}\sum_{\mu}\operatorname*{res}_{\zeta=q_\mu}\frac{1}{x-Q(\zeta)}\frac{1}{x'-Q(\zeta')}\frac{1}{x''-Q(\zeta'')}\Omega^{(3,1)}(\zeta,\zeta',\zeta'';\zeta) \tag{3.29}$$

which provides the result by residue evaluation (the only residues are at the three points $\zeta(x), \zeta(x'), \zeta(x''))$. The other correlators are computed exactly in the same way. "Q. E. D."

4. Conclusion and outlook

The main result of the paper is the formula for the triple derivatives of the free energy for arbitrary genus spectral curves. En route we have computed the variation of the conformal structure of the curve, which would in principle allow to compute derivatives of any order. Quite clearly the task becomes an exercise in complication (see similar problems in [26]). We also remark that in many practical uses the third derivatives suffices to determine many relevant properties. In particular our formulas should be sufficient to address completely the issue of associativity in both sets of variables $\{u_K\}$ and $\{v_J\}$. It should be remarked that the multi-loop correlators of the one-matrix model (which is a sub-case of the two-matrix model when one potential is gaussian) were presented by means of a recursive (and hence not totally explicit) procedure in [44].

We have not considered the mixed correlator

$$\mathfrak{M}(x,y) := \left\langle \mathrm{Tr}\,\frac{1}{x-M_1}\frac{1}{y-M_2}\right\rangle. \tag{4.1}$$

Such correlator is not universal and can be obtained via loop equations in the large N-limit [45, 18, 28, 46]. The mixed correlator (4.1) has also already been computed exactly in the finite $N = \dim(M_i)$ regime in [16].

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A. Example: the gaussian-gaussian model

If both potentials are gaussian ($d_1 = d_2 = 1$) then the spectral curve is *a fortiori* rational. There is nothing new in what we will write in this appendix since the tau-function has already been computed (see [30], with the warning that we are using different normalizations for the parameters) corresponding to the elliptic case (it suffices to consider the holomorphic and antiholomorphic coordinates as independent in their formulas to obtain ours). We choose the normalization of the coordinate λ so that

$$\begin{aligned} Q(\lambda) &= \gamma\lambda + \alpha_0 + \alpha_1\lambda^{-1} \\ P(\lambda) &= \gamma\lambda^{-1} + \beta_0 + \beta_1\lambda\,, \qquad \gamma \in \mathbb{R}_+\,. \end{aligned} \tag{A.1}$$

Using the formula for the free energy and computing the residues one obtains

$$2\mathcal{F}_0 = t^2 \ln\left(\frac{T}{\sqrt{u_2 v_2 - 1}}\right) - \frac{3}{2}t^2 + t\frac{(u_2 {v_1}^2 + v_2 u_1^2 + 2v_1 u_1)}{u_2 v_2 - 1}\,. \tag{A.2}$$

In this case one can check by direct computations that our residue formulas yield correctly the third derivatives of $\mathcal{F}_0$ (a computer aided algebra system like Maple will help for the computations of residues).

B. Relation with conformal maps

We want here to show that the "residue" formulas in [26, 32] are contained in our formulas. We recall that the comparison is achieved by taking (see section 2) $z = Q$, $d_1 = d_2 = d$, $V_1 = \overline{V_2} = V$ and using

$$\begin{aligned} Q(\lambda) &= \gamma\lambda + \sum_{i=0}^{d} \alpha_j \lambda^{-i} \\ P(\lambda) &= \frac{\gamma}{\lambda} + \sum_{i=0}^{d} \overline{\alpha}_i \lambda^{i} \\ &\gamma \in \mathbb{R}_+\,. \end{aligned} \tag{B.1}$$

In this case the Bergman kernel is simply

$$\Omega(\lambda, \lambda') = \frac{\mathrm{d}\lambda \mathrm{d}\lambda'}{(\lambda - \lambda')^2}\,, \tag{B.2}$$

and the differentials of the second kind reduce to

$$\mathcal{U}_K = (\partial_{u_K} P)_Q \mathrm{d}Q = \frac{1}{K} \operatorname*{res}_{\lambda'=\infty} \Omega(\lambda, \lambda') Q(\lambda)^K = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(Q^K(\lambda)\right)_+ \mathrm{d}\lambda = \frac{1}{K}\mathrm{d}\left(Q^K(\lambda)\right)_+\,, \tag{B.3}$$

where the subscript $()_+$ means the nonnegative part in λ. Then our formulas reproduce exactly those in [32] when proper identifications are made (our normalization differs from theirs).

C. Example: the gaussian-cubic case

Here we show how to compute the free energy and derivatives for the case $V_2(y) = \frac{v_2}{2}y^2 + v_1 y$ and V_1 cubic (one could do the exercise for higher degree potentials, the expressions become quite large but can be obtained explicitly e.g. with Maple). This case corresponds to the one-matrix model[8] and indeed the spectral curve is hyperelliptic. We will however restrict ourselves to the genus zero case where computations can be made completely explicit. Indeed we have

$$\begin{aligned} Q(\lambda) &= \gamma\lambda + \alpha_0 + \frac{\alpha_1}{\lambda} \\ P(\lambda) &= \frac{\gamma}{\lambda} + \sum_{j=0}^{2} \beta_j \lambda^j \,. \end{aligned} \tag{C.1}$$

We see that here the computations of all derivatives are possible in explicit terms because Q has only two critical points; in all cases in which (when computing the third derivatives) we should sum over the residues at the critical points of P we can replace the sum by summing over the critical points of Q plus the other residue at $\lambda = 0$. We can eliminate the parameters α_i as follows (sometimes that makes formulas shorter)

$$\alpha_0 = v_1 + v_2\beta_0\,, \qquad \alpha_1 = v_2\gamma\,, \qquad t = \alpha_1\beta_1 - \gamma^2\,. \tag{C.2}$$

In order to express the free energy in terms of the coordinates u_k we should invert an algebraic set of equations which can be done numerically at least. Nonetheless we can compute the free energy and any derivative up to the third using our formulas. The quadri differential $\Omega^{3,1}$ is simply

$$\Omega^{(3,1)}(\lambda_1,\lambda_2,\lambda_3;\lambda) = \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3\mathrm{d}\lambda}{(\lambda_1-\lambda)^2(\lambda_2-\lambda)^2(\lambda_3-\lambda)^2P'(\lambda)Q'(\lambda)}\,. \tag{C.3}$$

Using e.g. Maple one can immediately compute all derivatives by residue evaluations. For instance the chemical potential is

$$\mu = \partial_t\mathcal{F}_0 = \gamma^2 - \alpha_0\beta_0 - \beta_1\alpha_1 + 2\,t\ln(\gamma) + \frac{4\,\beta_2\alpha_0\alpha_1 + {\alpha_0}^2\beta_1 + {\beta_0}^2\alpha_1}{2\gamma} - \frac{\beta_2{\alpha_0}^3}{3\gamma^2} \tag{C.4}$$

and we have (just two examples)

$$\partial_t^3\mathcal{F}_0 = \frac{-\gamma^3 + v_2\gamma^2\beta_1}{\gamma^5 - 4\,{\beta_2}^2{v_2}^3\gamma^3 + \gamma^3{v_2}^2{\beta_1}^2 - 2\,\gamma^4v_2\beta_1} \tag{C.5}$$

$$\begin{aligned} \partial_{v_2}^3\mathcal{F}_0 = 8\Big(\gamma\Big(&\gamma^2{\beta_1}^3 + 4\,\gamma^3{\beta_2}^2 + 12\,\gamma^2\beta_0\beta_1\beta_2 + 2\,\gamma\,{\beta_0}^3\beta_2 + 3\,\gamma\,{\beta_1}^2{\beta_0}^2 + 24\,{\beta_2}^3\gamma\,\beta_0{v_2}^2 + \\ &+ 4\,{\beta_1}^2{\beta_2}^2\gamma\,{v_2}^2 + 12\,\beta_1{\beta_2}^2{\beta_0}^2{v_2}^2 + 4\,\gamma^2\beta_1{\beta_2}^2v_2 + 12\,\gamma\,{\beta_0}^2{\beta_2}^2v_2 - \\ &- 6\,\gamma\,\beta_2\beta_0{\beta_1}^2v_2 - \gamma\,{\beta_1}^4v_2 - 3\,{\beta_1}^3{\beta_0}^2v_2\Big)\Big)\times \\ \times\left(-\gamma^2 + 4\,{\beta_2}^2{v_2}^3 - {\beta_1}^2{v_2}^2 + 2\,v_2\gamma\,\beta_1\right)^{-1}. \end{aligned} \tag{C.6}$$

[8] Quite clearly the corresponding matrix model would have to be defined with normal matrices with spectrum on certain contours, rather than hermitean matrices [13]

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